The Decomposition of Idempotents Associated with Inflators

Jeffrey L. Stuart
Department of Mathematics
University of Southern Mississippi
Hattiesburg, Mississippi 39406

Submitted by Hans Schneider

ABSTRACT

In a recent paper, Friedland, Hershkowitz, and Schneider introduced a new matrix product called the inflation product and a new class of matrices called inflators. Fundamental to the constructions were certain idempotent matrices associated with the inflators. This paper studies the structure of the idempotent matrix associated with an inflator. In particular, it is shown that if the idempotent associated with an inflator has rank greater than one, then the idempotent can be split into several pairwise orthogonal idempotents of lower rank such that the resultant idempotents are associated with inflators which are inflation product factors of the original inflator. The indecomposable idempotents associated with the decomposition of an inflator are characterized in terms of rank, and are shown to be generally nonunique. The number of indecomposable idempotents in a splitting is shown to be invariant.

1. INTRODUCTION

In a recent paper [1], Friedland, Hershkowitz, and Schneider introduced the concept of the inflation product. The inflation product was used by those authors to construct the members of a certain subclass of Z-matrices called the ZME-matrices. The ZME-matrices, which are matrices all of whose positive powers are Z-matrices, all of whose odd powers are irreducible, and all of whose even powers are completely reducible, were shown to be diagonalizable matrices with real spectra whose spectral projectors were certain pairwise orthogonal, idempotent matrices. Each such idempotent was associated with a special type of matrix called an inflator. Consequently, the fundamental building blocks in the constructions in [1] are inflators. Inflators have been the subject of several recent papers [2]–[5].

This paper studies the structure of the idempotent matrix associated with an inflator. In particular, it is shown that if the idempotent associated with an inflator has rank greater than one, then the idempotent can be split into several pairwise orthogonal idempotents of lower rank such that the resultant idempotents are associated with inflators which are inflation product factors of the original inflator. This process of splitting an idempotent associated with an inflator is called decomposition. In relation to *ZME*-matrices, decomposition plays a key role in the study of spectral perturbations which preserve membership in the class of *ZME*-matrices [2, 3]. It will be shown in this paper that the idecomposable idempotents are precisely those of rank one. It will also be shown that the resultant idempotents from a decomposition are not generally unique, and that the only invariant for a decomposition of an idempotent associated with an inflator into indecomposable idempotents associated with inflators is the total number of idecomposable idempotents in such a splitting, and that number is exactly the rank of the original idempotent. It is useful to consider the example in Section 3.

The principal results of this paper are Theorems 4.1 and 4.2, which are the fundamental decomposition theorems, and Theorem 5.2, which relates decomposition to inflation sequences.

2. BASIC DEFINITIONS AND RESULTS

Throughout this paper, $\mathcal{M}_{m,n}(\mathcal{F})$ will be the set of all $m \times n$ matrices over the set $\mathcal{F}$, where $\mathcal{F}$ is either $\mathbb{R}$ or $\mathbb{C}$. The *support* of a matrix A in $\mathcal{M}_{m,n}(\mathcal{F})$ is the set $\{(i, j): a_{ij} \neq 0\}$. Two matrices are said to have *nonoverlapping support* if their supports have empty intersection. Two square matrices are said to be *orthogonal* if their product is the zero matrix. The set of $1 \times n$ matrices over $\mathcal{F}$ will be denoted as $\mathcal{F}^n$, and the term "vector" will always mean row vector. A *strictly nonzero matrix* (*strictly nonzero vector*) will be a matrix (vector) each of whose entries is nonzero. A *strictly positive matrix* (*strictly positive vector*) will be a real matrix (vector) each of whose entries is positive.

Let m and n be positive integers with $m \leqslant n$. An *m-partition of n* is a partition of the set $\{1, 2, \ldots, n\}$ into an ordered collection of m nonempty, disjoint sets such that the elements within each set are arranged in ascending order.

Throughout this paper, the following conventions are assumed: First, m and n are positive integers with $m \leqslant n$; and second, the set Π is an m-partition of n given by $B_1, B_2, \ldots, B_m$.

Let U be in $\mathcal{M}_{n,n}(\mathbb{C})$. The partition Π induces a block partitioning of the matrix U as follows: For $1 \leqslant i, j \leqslant m$, the i, j block of U consists of all entries $U_{\alpha\beta}$ such that α is in B_i and β is in B_j. Denote the i, j block of U by $U_{\langle i,j\rangle}$.

Let x be in $\mathbb{C}^n$. Then Π partitions x into m subvectors such that the ith subvector has entries x_α, where $\alpha \in B_i$. Denote the ith subvector by $x_{\langle i\rangle}$.

Let A be in $\mathscr{M}_{m,m}(\mathbb{C})$. Let U be in $\mathscr{M}_{n,n}(\mathbb{C})$. The *inflation matrix of A by U with respect to the partition* Π is the $n \times n$ matrix denoted by $A \times\times U$, which is defined as follows: For each α and β in $\{1,2,\dots,n\}$, there exist unique indices r and s such that $\alpha \in B_r$ and $\beta \in B_s$; let $(A \times\times U)_{\alpha\beta} = a_{rs}U_{\alpha\beta}$. Equivalently, in the block partition induced by the partition Π, $(A \times\times U)_{\langle r,s\rangle} = a_{rs}U_{\langle r,s\rangle}$ for each r and s. When the partitions are clear, $A \times\times U$ will be called *A inflated by U*. This is the definition of inflation given in [1, Definition 4.1].

If u is in $\mathbb{C}^m$ and v is in $\mathbb{C}^n$, then for each i in $\{1,2,\dots,m\}$, let $(u \times\times v)$ be the vector in $\mathbb{C}^n$ defined by $(u \times\times v)_{\langle j\rangle} = u_i v_{\langle i\rangle}$ for each i, where the partitioning is with respect to Π. Then $u \times\times v$ is called the *inflation vector of u by v with respect to the partition* Π.

The following theorem is stated in [1] for matrices, and an explicit proof can be found in [4] for both vectors and matrices.

THEOREM 2.1 (Associativity of inflation). *Let p be a positive integer such that $n \leqslant p$. Let Ω be an n-partition of p. Let A be in $\mathscr{M}_{m,m}(\mathbb{C})$. Let U be in $\mathscr{M}_{n,n}(\mathbb{C})$. Let V be in $\mathscr{M}_{p,p}(\mathbb{C})$. Then there exists an m-partition Γ of p such that*

$$(A \times\times U) \times\times V = A \times\times (U \times\times V).$$

Further, if a is in $\mathbb{C}^m$, u is in $\mathbb{C}^n$, and v is in $\mathbb{C}^p$, then

$$(a \times\times u) \times\times v = a \times\times (u \times\times v),$$

where the partition on the right is Γ.

Let U be in $\mathscr{M}_{n,n}(\mathbb{C})$. The matrix U is called an *inflator* (*with respect to* Π) if there exist vectors u and $\hat{u}$ in $\mathbb{C}^n$ which are partitioned by Π such that the following conditions hold:

(i) u and $\hat{u}$ are strictly nonzero vectors.
(ii) For $1 \leqslant i, j \leqslant m$, $U_{\langle i,j\rangle} = [u_{\langle i\rangle}]^t[\hat{u}_{\langle j\rangle}]$.
(iii) For $1 \leqslant i \leqslant m$, $u_{\langle i\rangle}[\hat{u}_{\langle u\rangle}]^t = 1$.

The pair of vectors u and $\hat{u}$ is called a *generating pair for the inflator U*. The matrix U is called a *normalized inflator* if u and $\hat{u}$ can be chosen so that they also satisfy a fourth condition:

(iv) For $1 \leqslant i \leqslant m$, $u_{\langle i\rangle}[u_{\langle i\rangle}]^* = \hat{u}_{\langle i\rangle}[\hat{u}_{\langle i\rangle}]^*$.

Observe that $U = u^t[\hat{u}]$. (These conditions are Definition 4.3 of [1].)

LEMMA 2.2. *Let U be an inflator with respect to Π. Then U is a strictly nonzero matrix, and the following hold:*

(i) *For $1 \leqslant i, j, k \leqslant m$, $U_{\langle i,j\rangle}U_{\langle j,k\rangle} = U_{\langle i,k\rangle}$.*
(ii) *For $1 \leqslant i \leqslant m$, $U_{\langle i,i\rangle}$ is an irreducible, idempotent matrix of rank one.*
(iii) *For all matrices A and B in $\mathscr{M}_{m,m}(\mathbb{C})$, $(A \times\times U)(B \times\times U) = (AB) \times\times U$.*
(iv) *For each matrix A in $\mathscr{M}_{m,m}(\mathbb{C})$,* $\operatorname{rank}(A \times\times U) = \operatorname{rank} A$.

Proof. See [1, section 4]. ■

THEOREM 2.3. *Let p be a positive integer with $n \leqslant p$. Let Ω be an n-partition of p given by the ordered collection of sets $C_1, C_2, \ldots, C_n$. Let Γ be the m-partition of p derived from Π and Ω. Then Γ is given by the ordered collection of sets $D_1, D_2,, \ldots, D_m$ where*

$$D_i = \bigcup_{j \in B_i} C_j.$$

Let U in $\mathscr{M}_{n,n}(\mathbb{C})$ be an inflator associated with Π, and let u and $\hat{u}$ be a generating pair for U. Let V in $\mathscr{M}_{p,p}(\mathbb{C})$ be an inflator associated with Ω, and let v and $\hat{v}$ be a generating pair for V. Let $W = U \times\times V$ with respect to Ω. Then W is an inflator associated with Γ, and W has generating pair $u \times\times v$ and $\hat{u} \times\times \hat{v}$.

Proof. See [4, Theorem 3.2]. ■

LEMMA 2.4. *Let U be an inflator. Suppose that U is a real matrix. Then U has a generating pair consisting of real vectors; and further, if U is normalized, then there is a generating pair of real vectors which satisfy the normalization condition.*

Proof. See [4, Lemma 3.6]. ■

Let U be an inflator associated with the m-partition Π of n. The matrix $G(U)$ is defined by $G(U) = I_n - (I_m \times\times U)$. Thus $G(U)$ can be expressed as the (internal) direct sum

$$G(U) = I_n - \left[\bigoplus_{i=1}^{m} U_{\langle i,i\rangle}\right] = \bigoplus_{i=1}^{m}\left[I - U_{\langle i,i\rangle}\right].$$

Thus $G(U)$ is permutation similar to a block-diagonal matrix.

LEMMA 2.5. *Let U be an inflator associated with the m-partition Π of n. Then $G(U)$ is an idempotent matrix of rank $n-m$. Further, each of the blocks $[I - U_{\langle i,i\rangle}]$ is an irreducible, idempotent matrix with nullity one such that every off-diagonal entry is nonzero. Finally, $B[G(U)] = [G(U)]B = 0$ for some B in $\mathcal{M}_{n,n}(\mathbb{C})$ if and only if $B = A \times\times U$ for some A in $\mathcal{M}_{m,m}(\mathbb{C})$.*

Proof. See [1, Section 4]. ∎

THEOREM 2.6. *Let p be a positive integer with $n \leqslant p$. Let Ω be an n-partition of p given by the ordered collection of sets $C_1, C_2, \dots, C_n$. Let Γ be the m-partition of p derived from Π and Ω as in Theorem 2.3. Let U in $\mathcal{M}_{n,n}(\mathbb{C})$ be an inflator associated with Π, and let V in $\mathcal{M}_{p,p}(\mathbb{C})$ be an inflator associated with Ω. Let $W = U \times\times V$ with respect to Ω. Then the inflator W associated with the partition Γ satisfies:*

(i) $G(W) = G(U) \times\times V + G(V)$,
(ii) $\operatorname{rank}[G(W)] = \operatorname{rank}[G(U)] + \operatorname{rank}[G(V)]$.

Proof. From the preceding theorem, W is an inflator associated with an m-partition of p. By Lemma 2.5, $\operatorname{rank}[G(W)] = p - m$, $\operatorname{rank}[G(U)] = n - m$, and $\operatorname{rank}[G(V)] = p - n$. Thus (ii) is immediate. It remains to show (i).

Throughout, the notation from Theorem 2.3 and its proof will be adopted. Thus the partition Π for U is expressed in terms of sets B_i, the partition Ω for V is expressed in terms of sets C_i, and the partition Γ for W is expressed in terms of sets D_i (with $d_i = |D_i|$ for each i). Blocks with respect to Γ will be denoted by $\langle\ ,\ \rangle$, while blocks with respect to Ω will be denoted by $\langle\ ,\ \rangle^*$.

Observe that

$$[G(W)]_{\langle i,j\rangle} = \begin{cases} 0 & \text{if } i \neq j, \\ I_{d_i} - W_{\langle i,i\rangle} & \text{if } i = j. \end{cases} \tag{2.7}$$

Since the block partitioning of V subpartitions the block partitioning of W, and since $[G(V)]_{\langle i,j\rangle^*}=0$ if $i\neq j$,

$$(2.8)\qquad [G(V)]_{\langle i,j\rangle}=\begin{cases}0 & \text{if}\quad i\neq j,\\ \bigoplus_{\alpha\in\beta_i}[G(V)]_{\langle\alpha,\alpha\rangle^*} & \text{if}\quad i=j.\end{cases}$$

Additionally, $[G(U)\times\times V]_{\langle i,j\rangle}$ subpartitions into blocks of the form

$$(2.9)\qquad [G(U)\times\times V]_{\langle\alpha,\beta\rangle^*}=[G(U)_{\alpha\beta}]V_{\langle\alpha,\beta\rangle^*}$$

for each α in B_i and each β in B_j. Since $G(U)_{\alpha\beta}=0$ unless $i=j$,

$$[G(U)\times\times V]_{\langle i,j\rangle}=0\qquad\text{if}\quad i\neq j.$$

Thus, in proving (i), it suffices to show that for $1\leqslant i\leqslant m$,

$$[G(W)]_{\langle i,i\rangle}=[G(U)\times\times V]_{\langle i,i\rangle}+[G(V)]_{\langle i,i\rangle}.$$

Fix i. Subpartition the $\langle i,i\rangle$ block into $\langle r,s\rangle^*$ blocks where $1\leqslant r,s\leqslant|B_i|$. It suffices to prove that for each r and s,

$$(2.10)\qquad [G(W)]_{\langle r,s\rangle^*}=[G(U)\times\times V]_{\langle r,s\rangle^*}+[G(V)]_{\langle r,s\rangle^*}.$$

First, suppose that $r=s$. Since W is an inflator, it has a generating pair w and $\hat{w}$ given by Theorem 2.3. Thus $W=w^t[\hat{w}]$. Then (2.7) becomes

$$I_{d_i}-[w_{\langle i\rangle}]^t[\hat{w}_{\langle i\rangle}].$$

Thus

$$[G(W)]_{\langle r,r\rangle^*}=I_{b_r}-[u_r\cdot v_{\langle r\rangle}]^t[\hat{u}_r\cdot\hat{v}_{\langle r\rangle}]=I_{b_r}-[u_r\hat{u}_r]\cdot V_{\langle r,r\rangle^*}.$$

The $\langle r,r\rangle^*$ subblock of $[G(U)\times\times V]$ is

$$[G(U)\times\times V]_{\langle r,r\rangle^*}=G(U)_{rr}\cdot V_{\langle r,r\rangle^*}=[1-u_r\hat{u}_r]\cdot V_{\langle r,r\rangle^*}.$$

Finally, the $\langle r,r\rangle^*$ subblock of $G(V)$ is

$$[G(V)]_{\langle r,r\rangle^*}=I_{b_r}-V_{\langle r,r\rangle^*}.$$

Clearly, (2.10) holds when $r=s$.

Suppose that $r \neq s$. Note that $G(V)_{\langle r,s\rangle^*} = 0$ is immediate from the definition of $G(V)$. Since $r \neq s$, the $\langle r,s\rangle^*$ subblock is an off-diagonal subblock. Thus

$$\begin{aligned}[G(W)]_{\langle r,s\rangle^*} &= -W_{\langle r,s\rangle^*} = -\left[u_r \cdot v_{\langle r\rangle}\right]^t\left[\hat{u}_s \cdot \hat{v}_{\langle s\rangle}\right]\\ &= -\left[u_r \hat{u}_s\right] \cdot V_{\langle r,s\rangle^*}.\end{aligned}$$

The $\langle r,s\rangle^*$ subblock of $G(U) \times\times V$ is

$$\left[G(U) \times\times V\right]_{\langle r,s\rangle^*} = G(U)_{rs} \cdot V_{\langle r,s\rangle^*} = \left[-u_r \hat{u}_s\right] \cdot V_{\langle r,s\rangle^*}.$$

Thus (2.10) holds when $r \neq s$. ∎

3. AN EXAMPLE OF MULTIPLE DECOMPOSITIONS

Consider the rank-three idempotent matrix

$$B = \frac{1}{75}\begin{bmatrix} 66 & -12 & -15 & -15 \\ -12 & 59 & -20 & -20 \\ -15 & -20 & 50 & -25 \\ -15 & -20 & -25 & 50 \end{bmatrix}.$$

Then $B = G(W)$, where W is the inflator with generating pair w and w, where

$$w = \frac{1}{5\sqrt{3}}(3 \quad 4 \quad 5 \quad 5).$$

Next, two decomposition of $G(W)$ will be exhibited. The first one is a decomposition into three idempotents, two with nonoverlapping support. The second one is a decomposition into three idempotents, all with overlapping supports.

Let $U_i = [u_i]^t[u_i]$ for $i = 1,2,3$, where $u_1 = (1/\sqrt{3})(1 \ \sqrt{2})$, $u_2 = (1/\sqrt{2})(\sqrt{2} \,|1 \ \ 1)$, and $u_3 = \frac{1}{5}(3 \ \ 4\,|5\,|5)$. Note that $w = u_1 \times\times u_2 \times\times u_3$. Thus $W = U_1 \times\times U_2 \times\times U_3$. Let $E_1 = G(U_1) \times\times U_2 \times\times U_3$, let $E_2 = G(U_2) \times\times U_3$, and let $E_3 = G(U_3)$. Then

$$E_1 = \frac{1}{150}\begin{bmatrix} 36 & 48 & -30 & -30 \\ 48 & 64 & -40 & -40 \\ -30 & -40 & 25 & 25 \\ -30 & -40 & 25 & 25 \end{bmatrix},$$

$$E_2 = [0] \oplus [0] \oplus \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

$$E_3 = \frac{1}{25}\begin{bmatrix} 16 & -12 \\ -12 & 9 \end{bmatrix} \oplus [0] \oplus [0].$$

Let $V_i = [v_i]^t[v_i]$ for $i = 1,2,3$, where $v_1 = (1/\sqrt{3})(\sqrt{2}\ \ 1)$, $v_2 = (1/\sqrt{2})(1\ \ 1 \,|\, \sqrt{2})$, and $v_3 = \frac{1}{5}(3\ \ 4 \,|\, 5 \,|\, 5)$. Note that $w = v_1 \times\times v_2 \times\times v_3$. Thus $W = V_1 \times\times V_2 \times\times V_3$. Let $F_1 = G(V_1) \times\times V_2 \times\times V_3$, let $F_2 = G(V_2) \times\times V_3$, and let $F_3 = G(V_3)$. Then

$$F_1 = \frac{1}{150}\begin{bmatrix} 36 & 48 & -30 & -30 \\ 48 & 64 & -40 & -40 \\ -30 & -40 & 25 & 25 \\ -30 & -40 & 25 & 25 \end{bmatrix},$$

$$F_2 = \frac{1}{50}\begin{bmatrix} 9 & 12 & -15 \\ 12 & 16 & -20 \\ -15 & -20 & 25 \end{bmatrix} \oplus [0],$$

$$F_3 = \frac{1}{25}\begin{bmatrix} 16 & -12 \\ -12 & 9 \end{bmatrix} \oplus [0] \oplus [0].$$

Finally, observe that $G(W) = E_1 + E_2 + E_3 = F_1 + F_2 + F_3$. Thus $G(W)$ has at least two different decompositions into rank-one idempotents associated with inflators. In fact, it is can be shown that these two decompositions are the only such decompositions for which the corresponding inflators are strictly positive matrices. However, it can also be shown that the inflators are not unique. In particular, there exists another pair of strictly positive inflators for the pair E_2 and E_3. (See [3, p. 85].)

4. DECOMPOSITION THEOREMS

THEOREM 4.1. *Let W be an inflator such that $G(W)$ has rank r for some $r \geq 2$. Then there exist inflators U and V such that $G(U)$ has rank one,*

$G(V)$ has rank $r-1$, and $W = U \times\times W$. If W is symmetric, then U and V may be chosen to be symmetric.

Proof. By the definition of an inflator, W has a generating pair consisting of strictly nonzero, partitioned vectors w and $\hat{w}$ such that $W = w^t\hat{w}$. In providing the existence of U and V, it suffices, by the definition of an inflator, to exhibit strictly nonzero, partitioned vectors u, $\hat{u}$, v, and $\hat{v}$ such that $W = [u^t\hat{u}] \times\times [v^t\hat{v}]$.

Suppose that W has block order m. Then the vectors w and $\hat{w}$ can be expressed as

$$w = \left(w_{\langle 1\rangle}\middle|w_{\langle 2\rangle}\middle|\cdots\middle|w_{\langle m\rangle}\right),$$

$$\hat{w} = \left(\hat{w}_{\langle 1\rangle}\middle|\hat{w}_{\langle 2\rangle}\middle|\cdots\middle|\hat{w}_{\langle m\rangle}\right).$$

Let s be the rank of the largest diagonal block of $G(W)$. Since the rank of $G(W)$ is r, which is greater than one, either $s \geqslant 2$, or else there are at least two rank-one diagonal blocks in $G(W)$. These two cases will be treated separately.

Suppose that inflators U and V exist such that $W = U \times\times V$. By Theorem 2.6, it follows that if $\operatorname{rank}[G(U)] = 1$, then

$$\operatorname{rank}[G(V)] = \operatorname{rank}[G(W)] - \operatorname{rank}[G(U)] = r - 1.$$

Case 1: Suppose that $s = 1$. Choose an index α such that $G(W)_{\langle\alpha,\alpha\rangle}$ is rank one. Then by Lemma 2.5, $W_{\langle\alpha,\alpha\rangle}$ is a 2×2 matrix. Thus $w_{\langle\alpha\rangle}$ has two elements. Since w and $\hat{w}$ are a generating pair for W, $w_{\langle\alpha\rangle} = (\nu_\alpha, \mu_\alpha)$ for some complex numbers ν_α and μ_α with $0 < |\nu_\alpha| < 1$ and $0 < |\mu_\alpha| < 1$. Similarly, $\hat{w}_{\langle\alpha\rangle} = (\hat{\nu}_\alpha, \hat{\mu}_\alpha)$, where the magnitudes of $\hat{\nu}_\alpha$ and $\hat{\mu}_\alpha$ are between zero and one, strictly. Let u and $\hat{u}$ be the following block-partitioned vectors, each with m blocks:

$$u_{\langle i\rangle} = \begin{cases} w_{\langle\alpha\rangle} & \text{if} \quad i = \alpha, \\ 1 & \text{if} \quad i \neq \alpha, \end{cases}$$

$$\hat{u}_{\langle i\rangle} = \begin{cases} \hat{w}_{\langle\alpha\rangle} & \text{if} \quad i = \alpha, \\ 1 & \text{if} \quad i \neq \alpha. \end{cases}$$

Since W is an inflator, $[w_{\langle\alpha\rangle}]^t\hat{w}_{\langle\alpha\rangle} = 1$, so $[u_{\langle i\rangle}]^t\hat{u}_{\langle i\rangle} = 1$ for $1 \leqslant i \leqslant m$. Clearly $u_{\langle i\rangle}$ and $\hat{u}_{\langle i\rangle}$ are strictly nonzero for each i. Thus $U = u^t\hat{u}$ is an inflator. If W is symmetric, then w and $\hat{w}$ can be chosen to be equal by Lemma 2.9 of [3], and then the symmetry of U is obvious. If W is normalized, then $w_{\langle\alpha\rangle}[w_{\langle\alpha\rangle}]^* = \hat{w}_{\langle\alpha\rangle}[\hat{w}_{\langle\alpha\rangle}]^*$; thus $u_{\langle i\rangle}[u_{\langle i\rangle}]^* = \hat{u}_{\langle i\rangle}[\hat{u}_{\langle i\rangle}]^*$ for each i. That is, U is normalized. Finally, $G(U)$ has a unique, nontrivial, diagonal block: the α,α block, which is 2×2; hence $\operatorname{rank}[G(U)] = 1$. Let v and $\hat{v}$ be the following block-partitioned vectors each with $m+1$ blocks (where blocks are denoted by $\langle\cdot\rangle^*$):

$$v_{\langle i\rangle^*} = \begin{cases} w_{\langle i\rangle} & \text{if } 1 \leqslant i < \alpha, \\ 1 & \text{if } i = \alpha, \alpha+1, \\ w_{\langle i-1\rangle} & \text{if } \alpha+1 < i \leqslant m+1, \end{cases}$$

$$\hat{v}^{(i)}_{\langle i\rangle^*} = \begin{cases} \hat{w}_{\langle i\rangle} & \text{if } 1 \leqslant i < \alpha, \\ 1 & \text{if } i = \alpha, \alpha+1, \\ \hat{w}_{\langle i-1\rangle} & \text{if } \alpha+1 < i \leqslant m+1. \end{cases}$$

Since W is an inflator, $[w_{\langle j\rangle}]^t\hat{w}_{\langle j\rangle} = 1$ for $1 \leqslant j \leqslant k$, so $[v_{\langle i\rangle^*}]^t\hat{v}_{\langle i\rangle^*} = 1$ for $1 \leqslant i \leqslant m+1$. Clearly $v_{\langle i\rangle^*}$ and $\hat{v}_{\langle i\rangle^*}$ are strictly nonzero for each i. Thus $V = v^t\hat{v}$ is an inflator. If W is symmetric, then it is clear that V is symmetric. If W is normalized, then for each i, $w_{\langle i\rangle}[w_{\langle i\rangle}]^* = \hat{w}_{\langle i\rangle}[\hat{w}_{\langle i\rangle}]^*$; thus $v_{\langle i\rangle^*}[v_{\langle i\rangle^*}]^* = \hat{v}_{\langle i\rangle^*}[\hat{v}_{\langle i\rangle^*}]^*$ for each i. That is, V is normalized. Finally, let $\langle\ ,\ \rangle$ denote the partitioning of W. Since

$$U \times\times V = [u \times\times v]^t[\hat{u} \times\times \hat{v}]$$

by Theorem 2.3, it suffices to prove that $w = u \times\times v$ and $\hat{w} = \hat{u} \times\times \hat{v}$. Consider $u \times\times v$ first. Then for $1 \leqslant j \leqslant m+1$,

$$[u \times\times v]_{\langle j\rangle^*} = u_j v_{\langle j\rangle^*} = \begin{cases} 1\cdot v_{\langle j\rangle^*} & \text{if } 1 \leqslant j < \alpha, \\ \nu_\alpha\cdot 1 & \text{if } j = \alpha, \\ \mu_\alpha\cdot 1 & \text{if } j = \alpha+1, \\ 1\cdot v_{\langle j-1\rangle^*} & \text{if } \alpha+1 < j \leqslant m+1. \end{cases}$$

Further, $u \times\times v$ repartitions into m blocks (with the partition Γ):

$$[u \times\times v]_{\langle i\rangle} = \begin{cases} w_{\langle i\rangle} & \text{if } 1 \leqslant i < \alpha, \\ (\nu_\alpha, \mu_\alpha) & \text{if } i = \alpha, \\ w_{\langle i\rangle} & \text{if } \alpha < i \leqslant m. \end{cases}$$

That is, $[u \times\times v]_{\langle i\rangle} = w_{\langle i\rangle}$ for $1 \leqslant j \leqslant m$. So $u \times\times v = w$. Similarly, $\hat{u} \times\times \hat{v} = \hat{w}$.

Case 2: Suppose that $s \geqslant 2$. Choose an index α such that $G(W)_{\langle\alpha,\alpha\rangle}$ is rank s. Then by Lemma 2.5, $W_{\langle\alpha,\alpha\rangle}$ is an $(s+1)\times(s+1)$ matrix. The subvectors $w_{\langle\alpha\rangle}$ and $\hat{w}_{\langle\alpha\rangle}$ each have $s+1$ entries. The magnitude of each entry of $w_{\langle\alpha\rangle}$ and $\hat{w}_{\langle\alpha\rangle}$ lies strictly between zero and one. Label the first entry of $w_{\langle\alpha\rangle}$ as ν, and label the first entry of $\hat{w}_{\langle\alpha\rangle}$ as $\hat{\nu}$. Since W is an inflator, $w_{\langle\alpha\rangle}[\hat{w}_{\langle\alpha\rangle}]^* = 1$. Let λ be the positive real number such that $\lambda^2 = 1 - \nu\hat{\nu}$. Then λ lies strictly between zero and one. Let u and $\hat{u}$ be the following block-partitioned vectors, each with m blocks:

$$u_{[i]} = \begin{cases} (\nu,\lambda) & \text{if} \quad i=\alpha, \\ 1 & \text{if} \quad i \neq \alpha, \end{cases}$$

$$\hat{u}_{[i]} = \begin{cases} (\hat{\nu},\lambda) & \text{if} \quad i=\alpha, \\ 1 & \text{if} \quad i \neq \alpha. \end{cases}$$

Since $\nu\hat{\nu} + \lambda^2 = 1$, $[u_{[i]}]^t\hat{u}_{[i]} = 1$ for $1 \leqslant i \leqslant m$. Clearly $u_{[i]}$ and $\hat{u}_{[i]}$ are strictly nonzero for each i. Thus $U = u^t\hat{u}$ is an inflator. If W is symmetric, then w and $\hat{w}$ can be chosen to be equal by Lemma 2.9 of [3]; hence $\nu = \hat{\nu}$, and the symmetry of U is obvious. It follows immediately from $\nu = \hat{\nu}$ that U is normalized. Finally, $G(U)$ has a unique, nontrivial, diagonal block: the α,α block, which is 2×2; hence $\operatorname{rank}[G(U)] = 1$.

Let y be the length-s vector formed from $w_{\langle\alpha\rangle}$ as follows: For $1 \leqslant i \leqslant s$, let $y_i = \lambda^{-1}[w_{\langle\alpha\rangle}]_{i+1}$. Then y is strictly nonzero. Similarly define the strictly nonzero vector $\hat{y}$ in terms of λ and $\hat{w}_{\langle\alpha\rangle}$. Let v and $\hat{v}$ be the following block-partitioned vectors, each with $m+1$ blocks:

$$v_{\langle\langle i\rangle\rangle} = \begin{cases} w_{\langle i\rangle} & \text{if} \quad 1 \leqslant i < \alpha, \\ 1 & \text{if} \quad i = \alpha, \\ y & \text{if} \quad i = \alpha+1, \\ w_{\langle i-1\rangle} & \text{if} \quad \alpha+1 < i \leqslant m+1, \end{cases}$$

$$\hat{v}_{\langle\langle i\rangle\rangle} = \begin{cases} \hat{w}_{\langle i\rangle} & \text{if} \quad 1 \leqslant i < \alpha, \\ 1 & \text{if} \quad i = \alpha, \\ \hat{y} & \text{if} \quad i = \alpha+1, \\ w_{\langle i-1\rangle} & \text{if} \quad \alpha+1 < i \leqslant m+1. \end{cases}$$

Note that $w_{\langle\alpha\rangle}[\hat{w}_{\langle\alpha\rangle}]^t = \nu\hat{\nu} + \Sigma_{i=2}^{s+1}[w_{\langle\alpha\rangle}]_i[\hat{w}_{\langle\alpha\rangle}]_i = \nu\hat{\nu} + [\lambda y][\lambda\hat{y}]^t$. Since W is an inflator, $w_{\langle\alpha\rangle}[\hat{w}_{\langle\alpha\rangle}]^t = 1$. Thus $1 = \nu\hat{\nu} + \lambda^2 \cdot y\hat{y}^t = (1-\lambda^2) + \lambda^2 \cdot y\hat{y}^t$. That is, $y\hat{y}^t = 1$. Thus $V = v^t\hat{v}$ is an inflator. By Lemma 2.9 of [3], if W is symmetric (hence normalized), then $w = \hat{w}$. Then $y = \hat{y}$; hence V is a symmetric (hence normalized) inflator.

From Theorem 2.3, $U \times\times V = [u \times\times v]^t[\hat{u} \times\times \hat{v}]$. It suffices to prove that $w = u \times\times v$ and $\hat{w} = \hat{u} \times\times \hat{v}$. Consider $u \times\times v$. This vector is naturally partitioned into $m+1$ blocks:

$$(u \times\times v)_{[j]} = \begin{cases} 1 \cdot w_{\langle j\rangle} & \text{if} \quad 1 \leqslant j < \alpha, \\ \nu \cdot 1 & \text{if} \quad j = \alpha, \\ \lambda \cdot y & \text{if} \quad j = \alpha + 1, \\ 1 \cdot w_{\langle j-1\rangle} & \text{if} \quad \alpha + 1 < j \leqslant m+1. \end{cases}$$

Observe that the vector formed by concatinating $\nu = [w_{\langle\alpha\rangle}]_1$ and λy is precisely $w_{\langle\alpha\rangle}$. Thus $u \times\times v$ repartitions into m blocks (denoted by $\langle\cdot\rangle$):

$$(u \times\times v)_{\langle j\rangle} = \begin{cases} w_{\langle j\rangle} & \text{if} \quad 1 \leqslant j < \alpha, \\ (\nu, \lambda y) & \text{if} \quad j = \alpha, \\ w_{\langle j\rangle} & \text{if} \quad \alpha < j \leqslant m. \end{cases}$$

That is, $(u \times\times v)_{\langle j\rangle} = w_{\langle j\rangle}$ for $1 \leqslant j \leqslant m$. So $w = u \times\times v$. Similarly, $\hat{w} = \hat{u} \times\times \hat{v}$. Thus $W = U \times\times V$. ∎

Remark. The proof of Theorem 4.1 can be modified [3, p. 92 ff.] to produce a more general result in the case that the original idempotent $G(W)$ has more than one diagonal block with rank greater than zero. Let $G(W)$ be of rank r with $r \geqslant 2$. Suppose that $G(W)$ has at least two nontrivial, diagonal blocks. Then W can be composed as $U \times\times V$ such that $G(U) \times\times V$ and $G(V)$ have nonoverlapping supports, and such that the nontrivial, diagonal blocks are arbitrarily distributed among $G(U) \times\times V$ and $G(V)$.

The following is an immediate consequence of Theorem 4.1.

Theorem 4.2 (The decomposition theorem). *Let W be a rank-r inflator for some $r \geqslant 2$. Then $G(W)$ can be decomposed into r pairwise orthogonal idempotents $P_1, P_2, \dots, P_r$ such that each P_i has rank one. That is, there exist r inflators $V_1, V_2, \dots, V_r$ such that $W = V_1 \times\times V_2 \times\times \cdots \times\times V_r$, such that $P_i = G(V_i) \times\times V_{i+1} \times\times \cdots \times\times V_r$ for each i, and such that $P_iP_j = \delta_{ij}P_i$ for all i and j. Further, if W is symmetric, then each V_i is normalized and symmetric, and hence each P_i is symmetric.*

REMARK. It follows immediately that the only indecomposable idempotents associated with inflators are the rank-one idempotents. Consequently, in a decomposition into indecomposable idempotents, the number of indecomposable idempotents is an invariant.

5. INFLATION SEQUENCES AND DECOMPOSITIONS

Let $n_0, n_1, n_2, \dots, n_k$ be a sequence of integers such that $n_0 = 0$ and $1 = n_1 < n_2 < \cdots < n_k = n$. For $1 < i \leqslant k$, let $P_{i-1,i}$ be an n_{i-1}-partition of n_i. Let $U_1 = [0]$, the 1×1 zero matrix. For $1 < i \leqslant k$, let U_i be an inflator assosiated with $P_{i-1,i}$. The sequence $\{U_i\}_{i=1}^{k}$ is called an *inflation sequence*. In each of the inflators U_i is normalized for $1 < i \leqslant k$, then the sequence is called a *normalized inflation sequence*.

If U is the 1×1 zero matrix, define $G(U)$ to be I_1, the 1×1 identity matrix.

If $\{U_i\}_{i=1}^{k}$ is an inflation sequence, we will adopt the convention that $G(U_i) \times\times U_{i+1} \times\times \cdots \times\times U_k = G(U_k)$ when $i = k$. For $1 \leqslant i \leqslant k$, let $E_i = G(U_i) \times\times U_{i+1} \times\times \cdots \times\times U_k$. Let $\mathscr{E}$ denote the set $\mathscr{E} = \{E_i : 1 \leqslant i \leqslant k\}$.

LEMMA 5.1. *Let $\{U_i\}_{i=1}^{k}$ be an inflation sequence. For $1 \leqslant i \leqslant k$, the $n \times n$ matrix E_i is an idempotent matrix of rank $n_i - n_{i-1}$. Further, the elements of $\mathscr{E}$ are pairwise orthogonal idempotents, and*

$$\sum_{i=1}^{k} E_i = I_n.$$

Proof. See [1, Section 6]. ∎

THEOREM 5.2. *Suppose that $\{U_i\}_{i=1}^{k}$ is an inflation sequence with corresponding set of pairwise orthogonal idempotents $\mathscr{E}$. Suppose that for some j, $G(U_j)$ has rank r with $r \geqslant 2$. Then there exist inflators $V_1, V_2, \dots, V_r$ such that $U_j = V_1 \times\times V_2 \times\times \cdots \times\times V_r$, and such that $G(V_i)$ is rank one for each i. Let $\{W_i\}_{i=1}^{k+r-1}$ be the sequence such that*

$$W_i = \begin{cases} U_i & \text{if } 1 \leqslant i < j, \\ V_{i-j+1} & \text{if } j \leqslant i < j + r, \\ U_{i-r+1} & \text{if } j + r \leqslant i \leqslant k + r - 1. \end{cases}$$

Then $\{W_i\}_{i=1}^{k+r-1}$ is an inflation sequence. Further, the corresponding set of *pairwise orthogonal idempotents is*

$$\mathscr{T} = \left[\mathscr{E}\backslash\{G(W_j) \times\times W_{j+1} \times\times \cdots \times\times W_k\}\right]$$
$$\cup \{G(V_i) \times\times \cdots \times\times V_r \times\times W_{j+1} \times\times \cdots \times\times W_k : 1 \leqslant i \leqslant r\}.$$

Proof. This follows immediately from Theorems 2.1, 2.3, and 4.2. ■

In [1], it is proven that an $n \times n$ matrix A is a *ZME*-matrix if and only if the distinct eigenvalues of A satisfy

$$-\alpha_2 \leqslant \alpha_1 < \alpha_2 < \cdots < \alpha_k,$$

and there exists a normalized inflation sequence $\{U_i\}_{i=1}^{k}$ such that U_i is strictly positive for $i \geqslant 2$ and such that

$$A = \sum_{i=1}^{k} \alpha_i E_i,$$

where the E_i are in $\mathscr{E}$. Consequently, decompositions of idempotents associated with the strictly positive inflators play a major role in studying the spectral perturbation of *ZME*-matrices. Numerous results for this case have been established; see [2], [3].

The author would like to express his appreciation to Professor Hans Schneider at the University of Wisconsin, who supervised the dissertation from which this paper comes.

REFERENCES

1 S. Friedland, D. Hershkowitz, and H. Schneider, Matrices whose powers are *M*-matrices or *Z*-matrices, *Trans. Amer. Math. Soc.*, 300 (1987) 343–366.
2 H. Schneider and J. Stuart, Allowable spectral perturbations for *ZME*-matrices, submitted for publication.
3 J. Stuart, *ZM*- and *MM*-matrices, Ph.D. Dissertation, Univ. of Wisconsin, Madison, 1986.
4 ____, Eigenvectors for inflation matrices and inflation-generated matrices, *Linear Multilinear Algebra*, to appear.
5 ____, Inflation matrices and *ZME*-matrices which commute with permutation matrices, *SIAM J. Matrix Analysis*, to appear.